Signal-to-Noise Estimates for Various Chop/Nod Techniques with FORCAST

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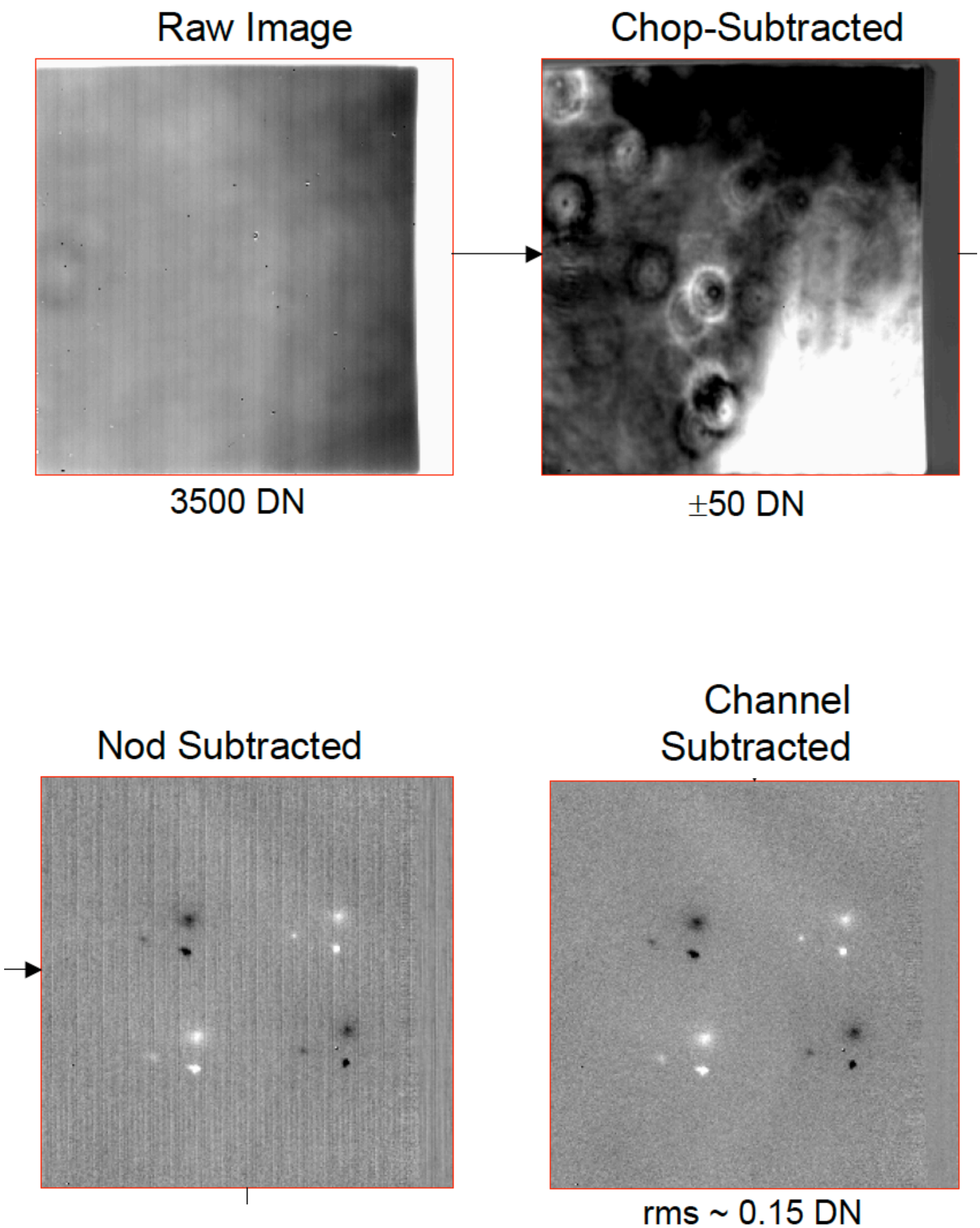
Introduction

Because the sky is highly temporally variable at MIR wavelengths, the common way to observe is to chop rapidly, by tilting the secondary, between the target object and a sky region. By subtracting the two frames at the two chop positions, the sky and telescope backgrounds can be eliminated. However, tilting the secondary results in small differences in the optical path of the beam between the two observations and produces a small, so-called 'radiative offset' between the two frames. Although the radiative offset is generally very small compared to the sky and telescope backgrounds, it can often be many times stronger than the signal from the source being observed. To remove this offset, it is typically necessary to nod the telescope to a new position and repeat the chop pattern. The two chop positions at the new nod position are subtracted and then the two chop-subtracted frames at the two nod positions are then subtracted from one another. This results in a cleaner image of the source, with much lower background and noise. An example of this procedure is shown on the next page, taken from Terry Herter's Nov. 2006 SITR presentation and displaying some FORCAST images obtained at Palomar.

For compact sources (i.e., much smaller than the size of the array), it possible to chop and nod such that the target source always remains on the array. For extended sources, the chop is usually set to a sky position that does not include the target (i.e., the target is not present on the array. However, 'off-chip' chopping can be used for compact sources as well.

The question arises as to what is the 'best' way to chop and nod, in terms of the resulting measured signal-to-noise, for a compact source. Below I present calculations for four example chop/nod scenarios under the assumption of background-limited conditions. These chop/nod scenarios are: (1) on-chip chopping, with a anti-parallel nod; (2) off-chip chopping with an parallel/anti-parallel nod; (3) on-chip chopping with a perpendicular nod; and (4) on-chip chopping with parallel nod. (Here, parallel and perpendicular refer to nod directions relative to the chop. The diagrams should make the nomenclature fairly clear.) The result of these calculations indicates that, counter-intuitively, **the first three chop/nod scenarios all yield the same S/N ratio.** The last chop/nod technique yields a S/N value that is a factor of $\sqrt{2}$ better than those of the other methods. Unfortunately this last technique is generally not available, as it requires the chop direction to switch within a given chop/nod cycle.

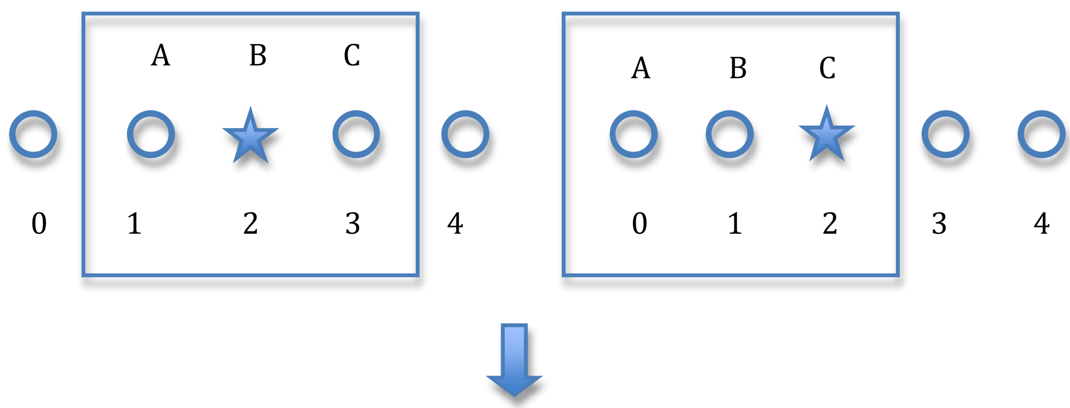
The calculations below are fairly straightforward, albeit tedious, algebra. I assume that the exposure times are the same for all frames, all flux values are given in electrons, and uncertainties obey Poisson statistics. For the final results, I assume that the observations are in the background-limited regime (which is always the case for the MIR). A simplified explanation that gives the same result for chop/nod scenarios (2) and (3) can be found on the FORCAST web page.



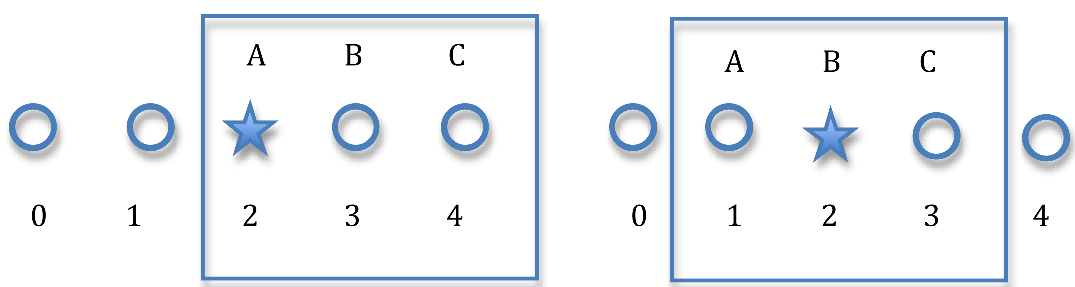
I. On-chip Chop with Anti-Parallel Nod

Consider the following observing scenario: A point source target is centered on the array for the first frame. The chop throw is set up to move the object to another position on the array for the "chopped" frame. (This is known as an asymmetric chop.) The telescope is then nodded to a new position, such that the object is now located on the opposite side of the array from where it was during the chop. The chop throw is kept the same as before, so that the chop now moves the target to the center of the array in the chopped frame. The exposure times at each chop and nod position are assumed to be identical in all of these examples. There are 3 positions on the array (A-C), corresponding to 5 sky positions (0-4), that must be considered.

Chop: Position 1 Position 2



Nod in the opposite direction of the chop (anti-parallel), then chop as before:



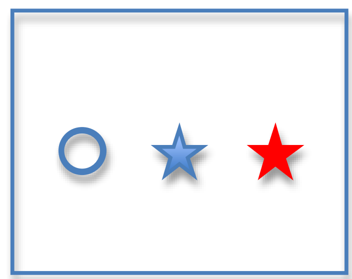
We consider the fluxes at points A, B, and C on the array. The fluxes have contributions from the Source (S), Background (B), Telescope (T), and the radiative offset term (O) resulting from the chopper position. All flux values are in electrons and we assume Poisson statistics. For the moment, we assume that each sky position has its own background flux. The telescope background T and the radiative offsets O are a functions only of the array positions. We do not consider changes in the PSF and image quality due to chopping.

For the first chop position, the three array positions have the following fluxes:
$F_A = B_1 + T_A$
$F_B = S + B_2 + T_B$
$F_C = B_3 + T_C$

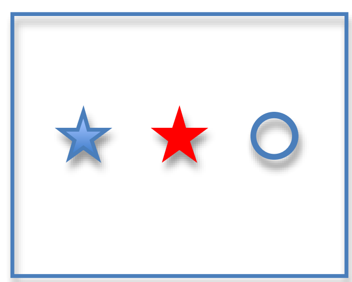
For the second chop position, the array positions have the following fluxes:
$F_A = B_0 + T_A + O_A$
$F_B = B_1 + T_B + O_B$
$F_C = S + B_2 + T_C + O_C$

After the nod, the first chop position gives the following fluxes on the array:
$F_A = S + B_2 + T_A$
$F_B = B_3 + T_B$
$F_C = B_4 + T_C$

The second chop position gives:
$F_A = B_1 + T_A + O_A$
$F_B = S + B_2 + T_B + O_B$
$F_C = B_3 + T_C + O_C$



Subtracting the two chop positions at the first nod position gives:
$F_{A,chop} = B_1 - B_0 + T_A - T_A - O_A$
$F_{B,chop} = S + B_2 - B_1 + T_B - T_B - O_B$
$F_{C,chop} = B_3 - B_2 - S + T_C - T_C - O_C$
and variances of:
$\sigma_{A,chop}^2 = B0 + B_1 + 2T_A + O_A$
$\sigma_{B,chop}^2 = S + B_1 + B_2 + 2T_B + O_B$
$\sigma_{C,chop}^2 = S + B_2 + B_3 + 2T_C + O_C$



Subtracting the two chop positions at the second nod position gives:
$F_{A,chop} = S + B_2 - B_1 + T_A - T_A - O_A$
$F_{B,chop} = B_3 - B_2 - S + T_B - T_B - O_B$
$F_{C,chop} = B_4 - B_3 + T_C - T_C - O_C$
and
$\sigma_{A,chop}^2 = S + B_1 + B_2 + 2T_A + O_A$
$\sigma_{B,chop}^2 = S + B_2 + B_3 + 2T_B + O_B$
$\sigma_{C,chop}^2 = B_3 + B_4 + 2T_C + O_C$

And now subtracting the two nods gives:
$F_{A,net} = -S + 2B_1 - (B_0 + B_2)$
$F_{B,net} = 2S + 2B_2 - (B_1 + B_3)$
$F_{C,net} = -S + 2B_3 - (B_2 + B_4)$

The variance in the flux at the three positions is given by the following:
$\sigma_{A,net}^2 = S + B_0 + 2B_1 + B_2 + 4T_A + 2O_A$
$\sigma_{B,net}^2 = 2S + B_1 + 2B_2 + B_3 + 4T_B + 2O_B$
$\sigma_{C,net}^2 = S + B_2 + 2B_3 + B_4 + 4T_C + 2O_C$
where we have assumed Poisson statistics, so that $\sigma^2 = F$.

We now have 3 estimates of the signal, one of which has double the exposure time of the others. The measured signal can be obtained from a weighted average of the three:
$<S> = (-F_{A,net} + 0.5F_{B,net} - F_{C,net})/3 = S + (B_0 - 2.5B_1 + 3B_2 - 2.5B_3 + B_4)/3$

The variance on the estimated signal is then given by:
$\sigma_{<S>}^2 = (\sigma_{A,net}^2 + 0.25\sigma_{B,net}^2 + \sigma_{C,net}^2)/9 =$
$(2.5S + B_0 + 2.25B_1 + 2.5B_2 + 2.25B_3 + B_4 + 4T_A + T_B + 4T_C +$
$2O_A + 0.5O_B + 2O_C)/9$

Let's assume that $B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_4$ and similarly that $T_A \approx T_B \approx T_C$ and $O_A \approx O_B \approx O_C$. Then $<S> = S$ and we have for the variances:

$\sigma_{A,net}^2 = S + 4B + 4T + 2O$
$\sigma_{B,net}^2 = 2S + 4B + 4T + 2O$
$\sigma_{C,net}^2 = S + 4B + 4T + 2O$

The variance on the estimated signal is then given by:
$\sigma_{<S>}^2 = 0.28S + (B + T + 0.5O) \approx B + T + 0.5O$
where the last approximation is for the typical case where the backgrounds are much brighter than the source.

Then the signal noise is given by

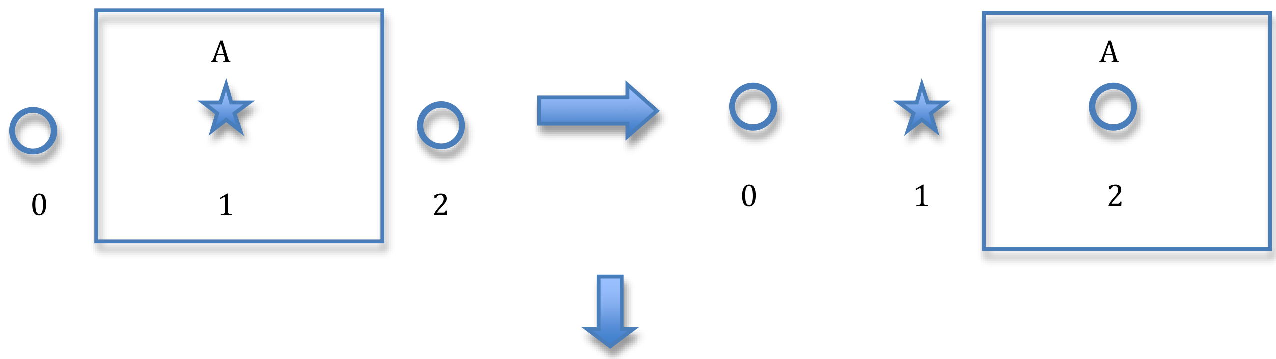
$S/N = S/(B + T + 0.5O)^{0.5}$.

Note that for any single chop pair, the noise is given by $\sigma = \sqrt{2}\,(B + T + 0.5O)^{0.5}$, as can be seen from the equations for F_{chop} above. This means that the final noise $\sigma_{<S>}$ is a factor of $\sqrt{2}$ smaller than that for any single chop pair. Note also that the S/N is not improved by considering the two negative images of the object. The S/N derived from the central image alone is the same as that derived from the average of the three images.

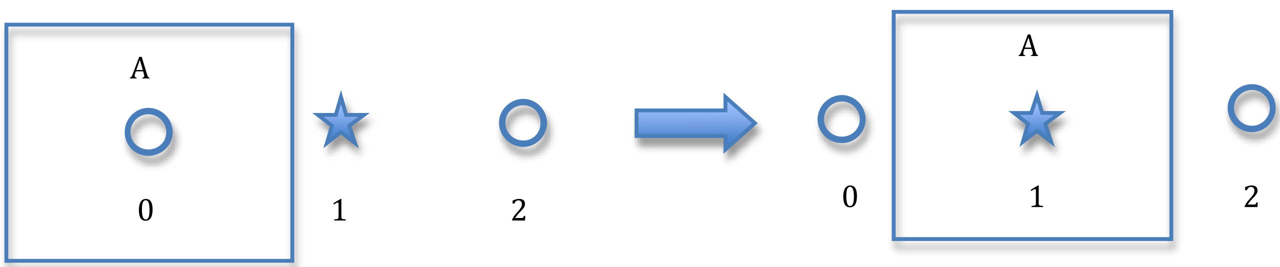
II. Off-chip Chop with Parallel/Anti-Parallel Nod

Now let us calculate the signal to noise ratio when we chop off source.

Chop: Position 1 Position 2



Nod in the opposite direction as the chop, then chop as before:



In this case, there is only a single array position (A) with which we are concerned, and three sky positions (0,1,2). In the first chop position, the flux at position A is given by:

$F_A = S + B_1 + T_A$

In the second chop position, the flux is:

$F_A = B_2 + T_A + O_A$

After the nod, the flux at position A in the first chop position is:

$F_A = B_0 + T_A$

And after the chop, the flux at position A in the second chop position is:

$F_A = S + B_1 + T_A + O_A$.

Subtracting the chop positions for the first nod position, we have:

$F_{A,chop} = S + B_1 - B_2 + T_A - T_A - O_A$

$\sigma_{A,chop}^2 = S + B_1 + B_2 + 2T_A + O_A$

The second nod position yields:
$F_{A,chop} = -S + B_0 - B_1 + T_A - T_A - O_A$
$\sigma_{A,chop}^2 = S + B_0 + B_1 + 2T_A + O_A$

Subtracting the two nods then gives
$F_{A,net} = 2S - B_0 + 2B_1 - B_2$

The variance on the flux from the first chopped pair minus the second chopped pair is given by:
$\sigma_{A,net}^2 = 2S + B_0 + 2B_1 + B_2 + 4T_A + 2O_A$

Now we have an estimate of the source flux with twice the exposure time, so the measured flux can be obtained by:
$\langle S \rangle = 0.5F_{A,net} = S + (2B_1 - B_0 - B_2)/2 \approx S$

So the variance on the estimated flux is given by:
$\sigma_{\langle S \rangle}^2 = 0.5S + 0.25(B_0 + 2B_1 + B_2) + T_A + 0.5O_A$

or, for the approximation above that $B_0 \approx B_1 \approx B_2$,
$\sigma_{\langle S \rangle}^2 = 0.5S + B + T + 0.5O \approx B + T + 0.5O$

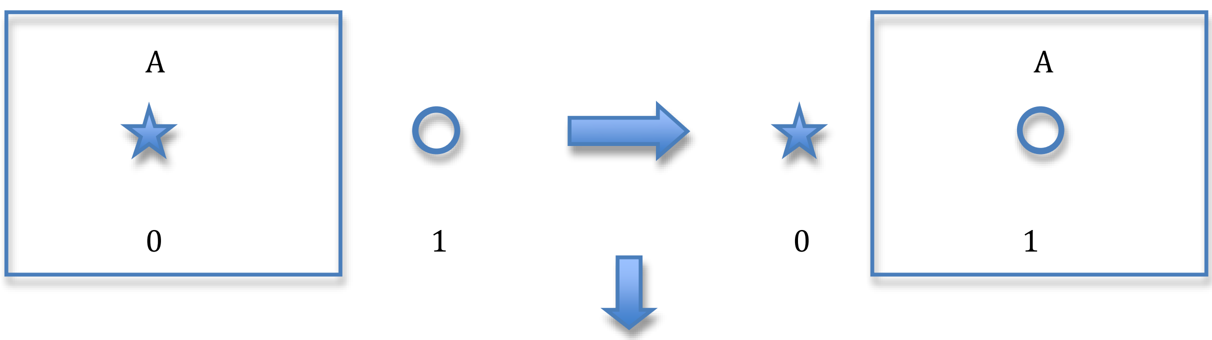
Hence the signal-to-noise ratio is then

$\mathbf{S/N = S/(B + T + 0.5O)^{0.5}}$

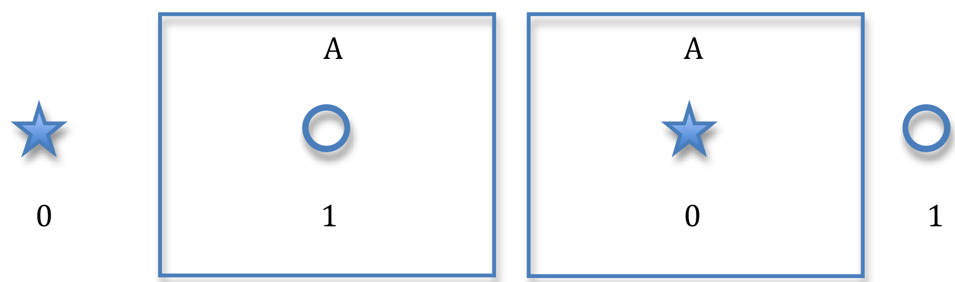
which is the same as for the on-chip chop/nod case. Again, the final noise is a factor of $\sqrt{2}$ smaller than that for a single chop pair.

Now suppose that instead of nodding in the opposite direction to the chop, we nodded in the same direction as the chop. In this case, we would have to switch the direction of the chop after the nod. (This is generally not possible, as it requires changing chop parameters in the middle of a chop/nod cycle. Nevertheless, we will follow through with the mathematics.) In this case we have:

Chop: Position 1 Position 2



Nod in the same direction as the chop, then chop in the opposite direction:



In this case, there is only a single array position (A) with which we are concerned, and two sky positions (0,1). In the first chop position, the flux at position A is given by:
$F_A = S + B_0 + T_A$

In the second chop position, the flux is:
$F_A = B_1 + T_A + O_A$

After the nod, the flux at position A in the first chop position is:
$F_A = B_1 + T_A$

And after the chop, the flux at position A in the second chop position is:
$F_A = S + B_0 + T_A + O_A'$,
where O_A does not necessarily equal O_A' because the chop is now in the opposite direction.

Subtracting the chop positions for the first nod position, we have:
$F_{A,chop} = S + B_0 - B_1 + T_A - T_A - O_A$
$\sigma_{A,chop}^2 = S + B_0 + B_1 + 2T_A + O_A$

The second nod position yields:
$F_{A,chop} = -S - B_0 + B_1 - T_A + T_A - O_A'$
$\sigma_{A,chop}^2 = S + B_0 + B_1 + 2T_A + O_A'$

Subtracting the two nods then gives
$F_{A,net} = 2S + 2B_0 - 2B_1 + O_A' - O_A = 2S + 2(B_0 - B_1) + (O_A' - O_A)$

The variance on the flux from the first chopped pair minus the second chopped pair is given by:
$\sigma_{A,net}^2 = 2S + 2B_0 + 2B_1 + 4T_A + O_A' + O_A$

Now we have an estimate of the source flux with twice the exposure time, so the measured flux can be obtained by:
$\langle S \rangle = 0.5F_{A,net} = S + (B_0 - B_1) + 0.5(O_A' - O_A) \approx S$

So the variance on the estimated flux is given by:
$\sigma_{\langle S \rangle}^2 = 0.5S + 0.5(B_0 + B_1) + T_A + 0.25(O_A' + O_A)$

or, for the approximation above that $B_0 \approx B_1$ and $O_0 \approx O_0'$,
$\sigma_{\langle S \rangle}^2 = 0.5S + B + T + 0.5O \approx B + T + 0.5O$

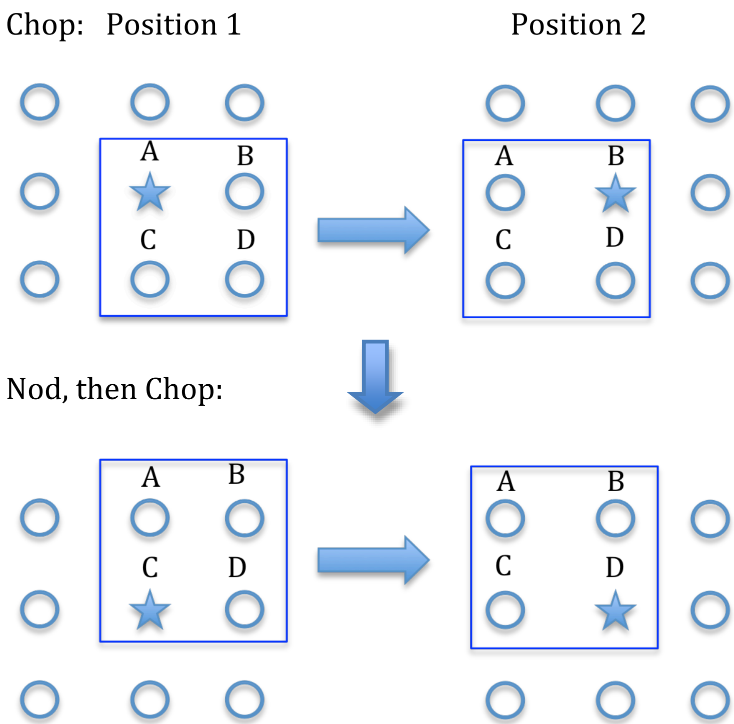
Hence the signal-to-noise ratio is then

$\mathbf{S/N = S/(B + T + 0.5O)^{0.5}}$

which is the same as for the on-chip chop/nod case. Again, the final noise is a factor of $\sqrt{2}$ smaller than that for a single chop pair.

III. On-Chip Chop with Perpendicular Nod

Now, let us consider the case of on-chip chopping with a perpendicular nod.



Now there are 4 array positions (A-D), corresponding to 9 sky positions (0-8, from the upper left to the lower right) that we must consider.

For the first chop position, we have

$F_A = S + B_4 + T_A$

$F_B = B_5 + T_B$

$F_C = B_7 + T_C$

$F_D = B_8 + T_D$

For the second chop position, we have:

$F_A = B_3 + T_A + O_A$

$F_B = S + B_4 + T_B + O_B$

$F_C = B_6 + T_C + O_C$

$F_D = B_7 + T_D + O_D$

For the first chop position after the nod, we have:

$F_A = B1 + T_A$

$F_B = B_2 + T_B$

$F_C = S + B_4 + T_C$

$F_D = B_5 + T_D$

For the second chop position after the nod, we have:

$F_A = B_0 + T_A + O_A$
$F_B = B_1 + T_B + O_B$
$F_C = B_3 + T_C + O_C$
$F_D = S + B_4 + T_D + O_D$

Again, proceeding as above, we subtract the two chop positions for the first nod:

$F_{A,chop} = S + B_4 - B_3 + T_A - T_A - O_A$
$F_{B,chop} = -S + B_5 - B_4 + T_B - T_B - O_B$
$F_{C,chop} = B_7 - B_6 + T_C - T_C - O_C$
$F_{D,chop} = B_8 - B_7 + T_D - T_D - O_D$

With variances given by:

$\sigma_{A,chop}^2 = S + B_4 + B_3 + 2T_A + O_A$
$\sigma_{B,chop}^2 = S + B_5 + B_4 + 2T_B + O_B$
$\sigma_{C,chop}^2 = B_7 + B_6 + 2T_C + O_C$
$\sigma_{D,chop}^2 = B_8 + B_7 + 2T_D + O_D$

Subtract the two positions for the second nod position:

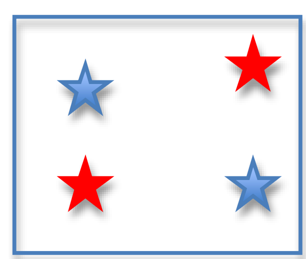
$F_{A,chop} = B_1 - B_0 + T_A - T_A - O_A$
$F_{B,chop} = B_2 - B_1 + T_B - T_B - O_B$
$F_{C,chop} = S + B_4 - B_3 + T_C - T_C - O_C$
$F_{D,chop} = -S + B_5 - B_4 + T_D - T_D - O_D$

With variances given by:

$\sigma_{A,chop}^2 = B_1 + B_0 + 2T_A + O_A$
$\sigma_{B,chop}^2 = B_2 + B_1 + 2T_B + O_B$
$\sigma_{C,chop}^2 = S + B_4 + B_3 + 2T_C + O_C$
$\sigma_{D,chop}^2 = S + B_5 + B_4 + 2T_D + O_D$

Now subtracting the two nods, we have:

$F_{A,net} = S + B_4 - B_3 - B_1 + B_0$
$F_{B,net} = -S + B_5 - B_4 - B_2 + B_1$
$F_{C,net} = -S + B_7 - B_6 - B_4 + B_3$
$F_{D,net} = S + B_8 - B_7 - B_5 + B_4$



and variances:

$\sigma_{A,net}^2 = S + B_0 + B_1 + B_3 + B_4 + 4T_A + 2O_A$
$\sigma_{B,net}^2 = S + B_1 + B_2 + B_4 + B_5 + 4T_B + 2O_B$
$\sigma_{C,net}^2 = S + B_3 + B_4 + B_6 + B_7 + 4T_C + 2O_C$
$\sigma_{D,net}^2 = S + B_4 + B_5 + B_7 + B_8 + 4T_D + 2O_D$

Now we have 4 estimates of the signal, and the measured flux can be obtained by averaging the four:

$<S> = (F_{A,net} - F_{B,net} - F_{C,net} + F_{D,net})/4 =$
$S + (B_0 - 2B_1 + B_2 - 2B_3 + 4B_4 - 2B_5 + B_6 - 2B_7 + B_8)/4$

and the variance is given by:

$\sigma_{<S>}^2 = 4S/16 + (B_0 + 2B_1 + B_2 + 2B_3 + 4B_4 + 2B_5 + B_6 + 2B_7 + B_8)/16 +$
$4(T_A + T_B + T_C + T_D)/16 + 2(O_A + O_B + O_C + O_D)/16$

For the approximations stated above about the backgrounds and radiative offsets, we have

$<S> = S$
$\sigma_{<S>}^2 \approx 0.25S + B + T + 0.5O \approx B + T + 0.5O$

The final signal-to-noise ratio is then:

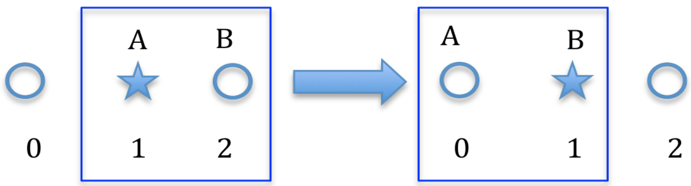
$S/N = S/(B + T + 0.5O)^{0.5}$

which is exactly the same as above. Again, the final noise is a factor of √2 smaller than that for a single chop pair.

IV. On-Chip Chop with Parallel Nod

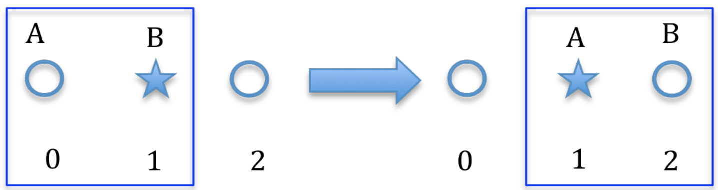
Now let us consider the case of on-chip chopping with a parallel nod:

Chop: Position 1 Position 2



Nod, then chop:

Position1 Position 2



In this case, there are two array positions (A,B) corresponding to 3 sky positions (0-2). The fluxes at the first chop position are:

$F_A = S + B_1 + T_A$
$F_B = B_2 + T_B$

And the fluxes at the second chop position are:
$F_A = B_0 + T_A + O_A$
$F_B = S + B_1 + T_B + O_B$

After the nod, the fluxes at the first chop position are:
$F_A = B_0 + T_A$
$F_B = S + B_1 + T_B$

The fluxes at the second chop position are:
$F_A = S + B_1 + T_A + O_A'$
$F_B = B_2 + T_B + O_B'$

Again, the primes indicate that the radiative offsets are not necessarily the same as before because the chop is in the opposite direction.

Subtracting the two chop positions for the first nod position, we have:
$F_{A,chop} = S + B_1 - B_0 + T_A - T_A - O_A$
$F_{B,chop} = -S + B_2 - B_1 + T_B - T_B - O_B$

The variances are:
$\sigma_{A,chop}^2 = S + B_0 + B_1 + 2T_A + O_A$
$\sigma_{B,chop}^2 = S + B_1 + B_2 + 2T_B + O_B$

Similarly, for the second chop position we have:
$F_{A,chop} = -S + B_0 - B_1 + T_A - T_A - O_A'$
$F_{B,chop} = S + B_1 - B_2 + T_B - T_B - O_B'$
$\sigma_{A,chop}^2 = S + B_0 + B_1 + 2T_A + O_A'$
$\sigma_{B,chop}^2 = S + B_1 + B_2 + 2T_B + O_B'$

Subtracting the two chopped-subtracted nod positions we have:
$F_{A,net} = 2S - 2B_0 + 2B_1$
$F_{B,net} = -2S - 2B_1 + 2B_2$
$\sigma_{A,net}^2 = 2S + 2B_0 + 2B_1 + 4T_A + O_A + O_A'$
$\sigma_{B,net}^2 = 2S + 2B_1 + 2B_2 + 4T_B + O_B + O_B'$

We now have 2 estimates of the source flux with double the exposure time, so the measured flux can be obtained by dividing the values by 2 and averaging them:
$<S> = (F_{A,net} - F_{B,net})/4 = S - (B_0 - 2B_1 + B_2)/2$
$\sigma_{<S>}^2 = (\sigma_{A,net}^2 + \sigma_{B,net}^2)/16 =$
$(4S + 2B_0 + 4B_1 + 2B_2 + 4T_A + 4T_B + O_A + O_A' + O_B + O_B')/16$

With the assumption that $B_0 \approx B_1 \approx B_2$ and $T_A \approx T_B$ and $O_A \approx O_A' \approx O_B \approx O_B'$, we have:
$<S> = S$
$\sigma_{<S>}^2 = 0.25S + 0.5B + 0.5T + 0.25O = 0.25S + 0.5(B + T + 0.5O) \approx 0.5(B + T + 0.5O)$

so the final S/N ratio is given by:

$S/N = \sqrt{2}S/(B + T + 0.5O)^{0.5}$

The final S/N value in this case is a factor of 2 larger than that for a single chop pair and a factor of √2 larger than that for the other three cases. Note, however, that this technique is generally not available at most telescopes, as it requires the chop direction to change within a given chop-and-nod cycle.